Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

sam.nitk.ac.in nitksam@gmail.com

MA110 - Engineering Mathematics-1
Problem Sheet - 2

Limits and Continuity in Higher Dimensions

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1. Whether the following limits exist or not. Justify your answer. Calculate the limit, if it exists.

   (a) $\lim\limits_{(x,y)\to(0,0)} \cos\left(\dfrac{x^2+y^3}{x+y+1}\right)$

   (b) $\lim\limits_{(x,y)\to(4,3)} \dfrac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$.

   (c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^2}{x^4+y^2}$

   (d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{|xy|}$

   (e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-xy^2}{x^2+y^2}$

   (f) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x}{x^2+x+y^2}$.

2. Let $f(x,y) = \dfrac{x+y}{x^2+1}$.

   (a) Let $\varepsilon = 0.001$. Find $\delta > 0$ such that $|f(x,y) - 0| < \varepsilon$ for every (x,y) with $\sqrt{x^2+y^2} < \delta$.
   (b) Let $\varepsilon = 0.001$. Find $\delta > 0$ such that $|f(x,y) - 1/2| < \varepsilon$ for all (x,y) with $\sqrt{(x-1)^2+y^2} < \delta$ [Hint: Try $\delta = \varepsilon$].
   (c) Show that $f(x,y)$ is continuous at $(1,0)$ by $\varepsilon - \delta$ definition.

3. Show that
$$\lim_{(x,y)\to(0,0)} (x+y) \neq 1$$
   using $\varepsilon - \delta$ definition.

4. Show that
$$\lim_{(x,y)\to(0,0)} y \sin\frac{1}{x} = 0.$$

5. Define $f(0,0)$ in a way that extends
$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$$
   to be continuous at the origin.

6. Find all the points where the functions are continuous.

   (a) $f(x,y) = \dfrac{x^2+y^2}{x^2-3x+2}$
   (b) $f(x,y) = \dfrac{1}{x^2-y}$
   (c) $f(x,y,z) = \dfrac{1}{|y|+|z|}$

7. Evaluate $\lim\limits_{(x,y)\to(1,1)} (x^2+3y)$.

8. Evaluate the following limits, whichever exist,

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

(b) $\lim\limits_{(x,y)\to(1,1)} \frac{x^2+y^2+2xy}{x+y}$

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2-2xy}{x-y}$

(d) $\lim\limits_{(x,y)\to(0,0)} \cos xy$

(e) $\lim\limits_{(x,y)\to(0,0)} \frac{xy\sin(xy)}{x^2+y^2}$

(f) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\sin\frac{1}{xy}$

(g) $\lim\limits_{(x,y)\to(0,0)} e^{xy}$

(h) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

9. Let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$ for $(x,y) \neq (0,0)$. Show that the iterated limits $\lim\limits_{y\to0}\left(\lim\limits_{x\to0} f(x,y)\right)$ and $\lim\limits_{x\to0}\left(\lim\limits_{y\to0} f(x,y)\right)$ exist, but $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

10. Show that the function f defined by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

is continuous at $(0,0)$.

11. Show that the function f defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

is continuous at $(0,0)$.

12. If $f(x,y) = \frac{x^2y^4}{(x^2+y^4)^2}$, then find

(a) $\lim\limits_{x\to0} f(x,mx)$

(b) $\lim\limits_{x\to0} f(x,\sqrt{x})$.

Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

13. Show that the following functions are discontinuous at origin:

(a) $f(x,y) = \begin{cases} \frac{1}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \frac{x^4-y^4}{x^4+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

(c) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

14. Show that the following functions are continuous at origin:

(a) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \frac{x^3y^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

15. Show that the following functions are discontinuous at $(0,0)$:

   (a) $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

   (b) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y, \\ 0, & x = y. \end{cases}$

   (c) $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

16. Discuss the following functions for continuity at $(0,0)$:

   (a) $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3}, & x^2+y^2 \neq 0, \\ 0, & x = y = 0. \end{cases}$

   (b) $f(x,y) = \begin{cases} 2xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

   (c) $f(x,y) = \begin{cases} 0, & (x,y) = (2y,y), \\ \exp\{|x-2y|/(x^2-4xy+4y^2)\}, & (x,y) \neq (2y,y). \end{cases}$

17. Show that f has a removable discontinuity at $(2,3)$.

   $f(x,y) = \begin{cases} 3xy, & (x,y) \neq (2,3), \\ 6, & (x,y) = (2,3). \end{cases}$

   Suitably redefine the function f to make it continuous.

18. Show that the function f is continuous at the origin, where

   $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

19. Can the given functions be appropriately defined at $(0,0)$ in order to be continuous there?

   (a) $f(x,y) = |x|^y$,

   (b) $f(x,y) = \sin\frac{x}{y}$,

   (c) $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$,

   (d) $f(x,y) = x^2\log(x^2+y^2)$.

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